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Application of Bohmian Mechanics to Dynamics of Prices of Shares: Stochastic Model of Bohm–Vigier from Properties of Price Trajectories

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Abstract We propose to describe behavioral financial factors (e.g., expectations of traders) by using the pilot wave (Bohmian) model of quantum mechanics. Through comparing properties of trajectories we come to the conclusion that the only possibility to proceed with real financial data is to apply the stochastic version of the pilot wave theory—the model of Bohm–Vigier.

Keywords Quantum-like models · Price-dynamics · Smoothness of trajectories · Quadratic variation of price-trajectories · Stochastic model of Bohm–Vigier

1 Introduction

Recently in [13] there was considered the application to finance of the stochastic version of the pilot wave theory—the model of Bohm–Vigier. We shall show that it seems that only such a stochastic model can provide an adequate description of the real financial data, cf. with applications to finance of the deterministic Bohmian model [3, 4].

Our paper can be considered as a contribution into applications of quantum mechanics to financial market, see [9–12, 16, 17, 21], see also [1, 5, 8, 16–19] on various applications of quantum mechanics outside the microworld. This paper is closely related to investigations on the *active information* interpretation of Bohmian mechanics [2, 14] and its applications to cognitive sciences, see also [17].

Our quantum-like model—financial Bohmian model—is a behavioral financial model. In our approach information about the financial market (including expectations of agents of the financial market) is described by an *information field* $\psi(q)$—*financial wave.* This field evolves deterministically[1] perturbing the dynamics of prices of stocks and options. Since

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[1] Dynamics is given by Schrödinger's equation on the space of prices of shares.

the psychology of agents of the financial market gives an important contribution into the financial wave $\psi(q)$, our model can be considered as a special *psycho-financial model.*

We remark also that there were performed investigations on the application of quantum methods to the financial market [9–12, 21] that were not directly coupled to behavioral modeling, but based on the general concept that randomness of the financial market can be better described by quantum mechanics.

We point out that there are two basic interpretations of quantum mechanics: *the quantum force interpretation* [2], and *the guidance-field interpretation*, e.g., [6, 7]. By the first the basic equation is the Newton equation for the position of a quantum particle and by the second—the guidance equation for its momentum.

It is interesting that the problem of interpretation can arise even in the financial framework. One of objections against applying the Bohmian quantum formalism for describing the dynamics of prices (of e.g. shares) of the financial market is smoothness of trajectories obtained in the Bohmian model—at least if one uses the *quantum force interpretation* [2]. In contrast to this, in financial mathematics it is commonly assumed that price-trajectories are not differentiable [20] or [22]. This is a problem.

This problem can be easily solved by using the guidance-field interpretation of Bohmian mechanics (so if one proceeds without introducing the quantum-like financial force [6]) and considering the integral version of the guidance equation. However, another objection can be presented even in this case. Solutions of the integral guidance equation have the zero quadratic variation. In contrast to this, in financial mathematics it is commonly assumed that price-trajectories have nonzero quadratic variations [20] or [22]. This is also a problem.

It seems that independently of the interpretation of Bohmian mechanics one could not apply it to financial market in the canonical deterministic form. The only possibility to proceed with real financial data is to apply the stochastic version of the pilot wave theory—the model of Bohm–Vigier.

2 Brief Introduction to Bohmian Mechanics

We emphasize that the conventional quantum formalism cannot say anything about the individual quantum particle. This formalism provides only statistical predictions for huge ensembles of particles. However, Bohmian mechanics [2, 6, 7, 15], provide a better description of quantum reality, since there is the possibility to describe trajectories of individual particles. This great advantage of Bohmian mechanics was not explored so much in physics. Up to now there have not been done experiments that would distinguish predictions of Bohmian mechanics and conventional quantum mechanics.

The mentioned advantages of Bohmian mechanics can be explored in applications to the financial market. In the latter case it is really possible to observe the trajectory of the price or price-change dynamics. Such a trajectory is described by equations of the mathematical formalism of Bohmian mechanics. The dynamics of the wave function $\psi(t,q)$ is described by Schrödinger's equation

$$ih\frac{\partial\psi}{\partial t}(t,q) = -\frac{h^2}{2m}\frac{\partial^2\psi}{\partial q^2}(t,q) + V(q)\psi(t,q). \tag{1}$$

Let us write the wave function $\psi(t,q)$ in the following form: $\psi(t,q) = R(t,q)e^{i\frac{S(t,q)}{h}}$, where $R(t,q) = |\psi(t,q)|$. We obtain the differential equations for R^2 and S:

$$\frac{\partial R^2}{\partial t} + \frac{1}{m}\frac{\partial}{\partial q}\left(R^2\frac{\partial S}{\partial q}\right) = 0. \tag{2}$$

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\left(V-\frac{h^2}{2mR}\frac{\partial^2 R}{\partial q^2}\right)=0. \tag{3}$$

We remark that if one uses the Born's probabilistic interpretation of the wave function, then $R^2=|\psi|^2$ gives the probability. Thus the equation (2) is the equation describing the dynamics of the probability distribution (in physics it is called the continuity equation). Suppose that $\frac{h^2}{2m}\ll 1$ and that the contribution of the term $\frac{h^2}{2mR}\frac{\partial^2 R}{\partial q^2}$ can be neglected. Then we obtain the equation:

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+V=0. \tag{4}$$

From the classical mechanics, we know that this is the classical Hamilton–Jacobi equation which corresponds to the dynamics of particles:

$$p=\frac{\partial S}{\partial q}$$

or

$$m\dot{q}=\frac{\partial S}{\partial q}, \tag{5}$$

where particles moves normal to the surface $S=\text{const}$. David Bohm proposed to interpret (3) in the same way. But we see that in this equation the classical potential V is perturbed by an additional "quantum potential" $U=\frac{h^2}{2mR}\frac{\partial^2 R}{\partial q^2}$.

From the evolution equation (5) we derive the usual Newton equation, but with the force corresponding to the combination of the classical potential V and the quantum one U:

$$m\frac{dv}{dt}=-\left(\frac{\partial V}{\partial q}-\frac{\partial U}{\partial q}\right). \tag{6}$$

The crucial point is that the potential U is by itself driven by a field equation—Schrödinger's equation (1). Thus (6) can not be considered as just the Newton classical dynamics (because the potential U depends on ψ as a field parameter).

We remark that there are two basic interpretations of quantum mechanics: *the quantum force interpretation* [2] and *the guidance-field interpretation*, see, e.g., [6, 7]. By the first the basic equation is the Newton equation (6) and by the second—the guidance equation (5). The first interpretation is closer to classical physics, because the basic equation is the second order Newton-like equation. But, as we have already mentioned, it is not just classical Newtonian mechanics, because there is also the field-type equation (Schrödinger's one) which is absent in classical physics. The second interpretation is rather far from classical mechanics, because even the notion of force is not involved into consideration. We would not like to be involved into debates on the right choice (if it is possible at all) of the interpretation of Bohmian mechanics. We will be very pragmatic and we would choose any interpretation which would be consistent with real financial data.

As we have already noted, there are a few critical arguments against the general Bohmian formalism:

1. Bohmian theory gives the possibility to provide the mathematical description of the trajectory $q(t)$ of an elementary particle. However, such a trajectory does not exist according to the conventional quantum formalism.

2. Bohmian theory is nonlocal, namely, via the pilot wave field one particle "feels" another on large distances.

We say that these disadvantages of theory will become advantages in our applications of Bohmian theory to financial market.

3 Price-Space, Classical Model

We repeat shortly the Bohmian model for the financial market which was proposed in [3, 4]. We start with the classical model for the price-dynamics. Let us consider a mathematical model in that a huge number of agents of the financial market interact with one another and take into account external economic (as well as political, social and even meteorological) conditions in order to determine the price to buy or sell financial assets. We consider the trade with shares of some corporations (e.g., VOLVO, SAAB, IKEA, ...).

We consider a *price system of coordinates.* We enumerate corporations which did emissions of shares at the financial market under consideration: $j = 1, 2, \dots, n$ (e.g., VOLVO: $j = 1$, SAAB: $j = 2$, IKEA: $j = 3$, ...). There can be introduced the n-dimensional configuration space $Q = R^n$ of prices, $q = (q_1, \dots, q_n)$, where q_j is the price of a share of the jth corporation. Here R is the real line. Dynamics of prices is described by the trajectory $q(t) = (q_1(t), \dots, q_n(t))$ in the configuration price space Q.

Another variable under the consideration is the *price change variable*: $v_j(t) = \dot{q}_j(t) = \lim_{\Delta t \to 0} \frac{q_j(t+\Delta t) - q_j(t)}{\Delta t}$, see, for example, the book Mantegna and Stanley (2000) on the role of the price change description. In real models we consider the discrete time scale $\Delta t, 2\Delta t, \dots$. Here we should use discrete price change variable $\delta q_j(t) = q_j(t + \Delta t) - q_j(t)$.

We now introduce an analogue m of mass as the number of items (in our case shares) that a trader emitted to the market. We call m the *financial mass.* Thus each trader j (e.g., VOLVO) has its own financial mass m_j (the size of the emission of its shares). The total price of the emission performed by the jth trader is equal to $T_j = m_j q_j$—*market capitalization.*

We also introduce *financial energy* of the market as a function $H : Q \times V \to R$. If we use the analogue with classical mechanics, then we could consider (at least for mathematical modeling) the financial energy of the form[2]: $H(q, v) = \frac{1}{2} \sum_{j=1}^{n} m_j v_j^2 + V(q_1, \dots, q_n)$.

Here $K = \frac{1}{2} \sum_{j=1}^{n} m_j v_j^2$ is the *kinetic financial energy* and $V(q_1, \dots, q_n)$ is the potential financial energy, m_j is the financial mass of jth trader.

The kinetic financial energy represents efforts of agents of financial market to change prices: higher price changes induce higher kinetic financial energies. If the corporation j_1 has higher financial mass than the corporation j_2, so $m_{j_1} > m_{j_2}$, then the same change of price, i.e., the same financial velocity $v_{j_1} = v_{j_2}$, is characterized by higher kinetic financial energy: $K_{j_1} > K_{j_2}$. We also remark that high kinetic financial energy characterizes rapid changes of the financial situation at market. However, the kinetic financial energy does not give the attitude of these changes. It could be rapid economic growth as well as recession.

The *potential financial energy* V describes the interactions between traders $j = 1, \dots, n$ (e.g., competition between NOKIA and ERICSSON) as well as external economic conditions (e.g., the price of oil and gas) and even meteorological conditions (e.g., the weather conditions in Louisiana and Florida). For example, we can consider the simplest interaction

[2] We are working just on the simulation level. The problem of finding of adequate forms of financial energy is of huge complexity!

potential: $V(q_1, \dots, q_n) = \sum_{j=1}^{n} (q_i - q_j)^2$. The difference $|q_1 - q_j|$ between prices is the most important condition for *arbitrage.*

As in classical mechanics for material objects, we introduce a new variable $p = mv$, the *price momentum* variable. Instead of the price change vector $v = (v_1, \dots, v_n)$, we consider the price momentum vector $p = (p_1, \dots, p_n)$, $p_j = m_j v_j$. The quantity $f_j(q) = -\frac{\partial V}{\partial q_j}$ is called the financial force. We can postulate the financial variant of the second Newton law:

$$m\dot{v} = f. \tag{7}$$

"The product of the financial mass and the price acceleration is equal to the financial force".

4 Bohmian Quantum-Like Model

Our fundamental assumption is that agents at the modern financial market are not just "classical-like agents". Their actions are ruled not only by classical-like financial potentials $V(t, q_1, \dots, q_n)$, but also (in the same way as in the pilot wave theory for quantum systems) by an additional information (or psychological) potential induced by a financial pilot wave.

Therefore we could not use the classical financial dynamics (Hamiltonian formalism) on the financial phase space to describe the real price trajectories. Information (psychological) perturbation of Hamiltonian equations for price and price change must be taken into account. To describe such a model mathematically, it is convenient to use such an object as a *financial pilot wave* that rules the financial market.

In some sense $\psi(q)$ describes the psychological influence of the price configuration q to behavior of agents of the financial market. In particular, the $\psi(q)$ contains expectations of agents.

We underline two important features of the financial pilot wave model:

1. All shares are coupled on the information level.
2. Reactions of the market do not depend on the amplitude of the financial pilot wave: waves ψ, 2ψ, 100000ψ will produce the same reaction.

5 Quantum Force Approach to Financial Market

By applying the quantum force approach to the financial market we postulate that the pilot wave (field) $\psi(q_1, \dots, q_n)$ induces a new (quantum) potential $U(q_1, \dots, q_n)$ which perturbs the classical equations of motion. A modified Newton equation has the form:

$$\dot{p} = f + g, \tag{8}$$

where $f = -\frac{\partial V}{\partial q}$ and $g = -\frac{\partial U}{\partial q}$. We call the additional financial force g a *financial mental force.* This force $g(q_1, \dots, q_n)$ determines a kind of collective consciousness of the financial market. Of course, the g depends on economic and other 'hard' conditions given by the financial potential $V(q_1, \dots, q_n)$. However, this is not a direct dependence. In principle, a nonzero financial mental force can be induced by the financial pilot wave ψ in the case of zero financial potential, $V \equiv 0$. So $V \equiv 0$ does not imply that $U \equiv 0$. *Market psychology is not totally determined by economic factors.* Financial (psychological) waves of information need not be generated by some changes in a real economic situation. They are mixtures of mental and economic waves. Even in the absence of economic waves, mental financial waves can have a large influence to the market.

By using the standard pilot wave formalism we obtain the following rule for computing the financial mental force. We represent the financial pilot wave $\psi(q)$ in the form: $\psi(q) = R(q)e^{iS(q)}$, where $R(q) = |\psi(q)|$ is the amplitude of $\psi(q)$ (the absolute value of the complex number $c = \psi(q)$) and $S(q)$ is the phase of $\psi(q)$ (the argument of the complex number $c = \psi(q)$). Then the financial mental potential is computed as

$$U(q_1, \dots, q_n) = -\frac{1}{R}\sum_{i=1}^{n} \frac{\partial^2 R}{\partial q_i^2}$$

and the financial mental force as $g_j(q_1, \dots, q_n) = \frac{-\partial U}{\partial q_j}(q_1, \dots, q_n)$. These formulas imply that strong financial effects are produced by financial waves having essential variations of amplitudes.

The only problem which we have still to solve is the description of the time-dynamics of the financial pilot wave, $\psi(t, q)$. We follow the standard pilot wave theory. Here $\psi(t, q)$ is found as the solution of Schrödinger's equation. Thus if we know $\psi(0, q)$ then by using Schrödinger's equation we can find the pilot wave at any instant of time t, $\psi(t, q)$. Then we compute the corresponding mental potential $U(t, q)$ and mental force $g(t, q)$ and solve Newton's equation. In this way we obtain the price-trajectory in our quantum-like model.

We have only to make one remark, namely, on the role of the constant h in Schrödinger's equation, cf. [9–13]. In quantum mechanics (which deals with microscopic objects) h is the Planck constant. This constant is assumed to play the fundamental role in all quantum considerations. However, originally h appeared as just a scaling numerical parameter for processes of energy exchange. Therefore in our financial model we can consider h as a price scaling parameter, namely, the unit in which we would like to measure price change. We do not present any special value for h. There are numerous investigations into price scaling. It may be that there can be recommended some special value for h related to the modern financial market, a *fundamental financial constant.* However, it seems that $h = h(t)$ evolves depending on economic development.

6 Problem of Smoothness of Price Trajectories

In the Bohmian model with the quantum force for price dynamics the price trajectory $q(t)$ can be found as the solution of the equation

$$m\frac{d^2q(t)}{dt^2} = f(t, q(t)) + g(t, q(t)) \tag{9}$$

with the initial condition $q(t_0) = q_0, q'(t_0) = q'_0$. Here we consider a "classical" (time dependent) force $f(t, q) = -\frac{\partial V(t,q)}{\partial q}$ and "quantum" force $g(t, q) = -\frac{\partial U(t,q)}{\partial q}$, where $U(t, q)$ is the quantum potential, induced by the Schrödinger dynamics. In Bohmian mechanics for *physical systems* (9) is considered as an ordinary differential equation and $q(t)$ as the unique solution (corresponding to the initial conditions $q(t_0) = q_0, q'(t_0) = q'_0$) of the class $C^2 : q(t)$ is assumed to be twice differentiable with continuous $q''(t)$. In contrast to it, in financial mathematics it is commonly assumed that the price-trajectory is not differentiable [20, 22].

7 Guidance-Field Approach to Financial Market

In this approach we should not go to the second order Newton-like equation for the dynamics of prices of shares. We postulate that prices evolves according to the first order equation (5). However, this differential equation can be the object of the same critique. It seems that the problem could be solved by considering the integral equation:

$$q(t) = q_0 + \frac{1}{m}\int_{t_0}^{t} \frac{\partial S}{\partial q} ds. \tag{10}$$

However, if the trajectory is continuous, then (for continuous guidance-field) this equation again implies that $q(t)$ is of the class C^1. One should consider a discontinuous guidance-field to get a nonsmooth trajectory. It is possible to find natural examples of discontinuous guidance-fields. However, there is an objection even against this model.

8 Quadratic Variation Objection

The quadratic variation of a trajectory $q(t)$ satisfying the integral equation (10) is equal zero. In contrast to this, in financial mathematics it is commonly assumed that the price-trajectory has nonzero quadratic variation [20, 22]. Therefore by moving from the quantum force to guidance-field interpretation one is not able to obtain trajectories which would match with real behavior of price trajectories. What can one do? It seems that the only possibility is use the Bohm–Vigier stochastic model.

9 Bohm–Vigier Stochastic Mechanics for Financial Market

The quadratic variation objection motivates consideration of the Bohm–Vigier stochastic model, instead of the completely deterministic Bohmian model. The basic assumption of Bohm and Vigier was that the velocity of an individual particle is given by

$$v = \frac{\nabla S(q)}{m} + \eta(t), \tag{11}$$

where $\eta(t)$ represents a random contribution to the velocity of that particle which fluctuates in a way that may be represented as a random process but with zero average. In Bohm–Vigier stochastic mechanics the quantum potential comes in through the average velocity and not the actual one.

We now shall apply the Bohm–Vigier model to financial market, see also [13]. Equation (11) is considered as the basic equation for the price velocity. Thus the real price becomes a random process (as well as in classical financial mathematics). We can write the stochastic differential equation, ..., for the price:

$$dq(t) = \frac{\nabla S(q)}{m} dt + \eta(t) dt. \tag{12}$$

To give the rigorous mathematical meaning to the stochastic differential we assume that

$$\eta(t) = \frac{d\xi(t)}{dt}, \tag{13}$$

for some stochastic process $\xi(t)$. Thus formally:

$$\eta(t)dt = \frac{d\xi(t)}{dt}dt = d\xi(t), \tag{14}$$

and the rigorous mathematical form of (12) is

$$dq(t) = \frac{\nabla S(q)}{m}dt + d\xi(t). \tag{15}$$

The expression (13) one can consider either formally or in the sense of distribution theory (we recall that for basic stochastic processes, e.g., the Wiener process, trajectories are not differentiable in the ordinary sense almost every where).

Suppose, for example, that the random contribution into the price dynamics is given by *white noise*, $\eta_{\text{white noise}}(t)$. It can be defined as the derivative (in sense of distribution theory) of the Wiener process:

$$\eta_{\text{white noise}}(t) = \frac{dw(t)}{dt},$$

thus:

$$v = \frac{\nabla S(q)}{m} + \eta_{\text{white noise}}(t). \tag{16}$$

In this case the price dynamics is given by the SDE:

$$dq(t) = \frac{\nabla S(q)}{m}dt + dw(t). \tag{17}$$

What is the main difference from the classical SDE-description of the financial market? This is the presence of the pilot wave $\psi(t,q)$, mental field of the financial market, which determines the coefficient of drift $\frac{\nabla S(q)}{m}$. Here $S \equiv S_\psi$. And the ψ-function is driven by a special field equation—Schrödinger's equation. The latter equation is not determined by the SDE (17). Thus, instead of one SDE, in the quantum-like model, we have the system of two equations:

$$dq(t) = \frac{\nabla S_\psi(q)}{m}dt + d\xi(t), \tag{18}$$

$$i\,h\frac{\partial\psi}{\partial t}(t,q) = -\frac{h^2}{2m}\frac{\partial^2\psi}{\partial q^2}(t,q) + V(q)\psi(t,q). \tag{19}$$

Finally we come back to the problem of the quadratic variation of the price. In the Bohm–Vigier stochastic model (for, e.g., the white noise fluctuations of the price velocity) is nonzero.

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